Classical solutions for the Carroll-Field-Jackiw-Proca electrodynamics

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In the present work, we investigate classical solutions of the Maxwell-Carroll-Field-Jackiw-Proca (MCFJP) electrodynamics for the cases a purely timelike and spacelike Lorentz-violating (LV) background. Starting from the MCFJP Lagrangian and the associated wave equations written for the potential four-vector, the tensor form of the Green function is achieved. In the timelike case, the components of the stationary Green function are explicitly written. The classical solutions for the electric and magnetic field strengths are then evaluated, being observed that the electric sector is not modified by the LV background, keeping the Maxwell-Proca behavior. The magnetic field associated with a charge in uniform motion presents an oscillating behavior that also provides an oscillating MCFJ solution (in the limit of a vanishing Proca mass), but does not recover the Maxwell-Proca solution in the limit of vanishing background. In the spacelike case, the stationary Green function is written and also explicitly carried out in the regime of a small background. The electric and magnetic fields reveal to possess an exponentially decaying behavior, that recover the Maxwell-Proca solutions.

I. INTRODUCTION

Lorentz covariance is regarded as one of the fundamental symmetries of nature. The rise and establishment of such a symmetry begun with the advent of Special Theory of Relativity, being after incorporated as a key feature of the modern Field Theories. Nowadays, Lorentz covariance pervades all known physical interactions, having the status of a cornerstone of modern physical theories. Investigations concerning LV are mostly conducted under the framework of the SME - Standard Model Extension, developed by Colladay & Kostelecky [1]. The Standard Model Extension (SME) is a broader version of the usual Standard Model that embraces all Lorentz-violating (LV) coefficients (generated as vacuum expectation values of tensor quantities belonging to an underlying theory at the Planck scale) that yield Lorentz scalars (as tensor contractions) in the observer frame. Such coefficients govern Lorentz violation in the particle frame, where they are seen as sets of independent numbers, whereas they work out as genuine tensor in the observer frame. One strong motivation to study the SME refers to the desire to get information on the Planck scale physics, where Lorentz violation may be allowed. This possible breaking is an important information for the development of quantum gravity theory. This kind of idea was supported by the demonstration concerning the possibility of Lorentz and CPT spontaneous breaking in the context of string theory [2]. In this scenario, a minuscule Lorentz violation at a lower energy scale (scrutinized into the framework of the SME) is to be read as a remanent effect of spontaneous Lorentz violation at Planck scale. Nowadays, Lorentz violation has been investigated in many different systems and purposes [3], involving also fermions [4], CPT and Lorentz-violating probing tests [5], topological phases [6], radiative corrections [7] and the gauge sector [8, 9].

Lorentz violation in the photon sector of the SME has been much investigated in literature with basically a twofold purpose: the determination of new electromagnetic effects induced by the Lorentz-violating coefficients and the imposition of stringent upper bounds on the LV coefficients that constrain the magnitude of Lorentz breaking. The pioneer investigation of LV effects on classical electromagnetism was performed by Carroll-Field-Jackiw [10], who studied the Maxwell electrodynamics in the presence of the assigned Carroll-Field-Jackiw term $\left(\epsilon^{\beta\alpha\rho\lambda}V_{\beta}A_{\alpha}F_{\rho\lambda}\right)$, with V_{β} standing for the LV fixed background. This Lorentz and CPT-odd term leads to a gauge invariant theory - Maxwell-Carroll-Field-Jackiw (MCFJ) electrodynamics, which is causal, stable and unitarity only for a purely spacelike background [11]. The photon sector of SME is composed by the Carroll-Field-Jackiw term and another Lorentz-violating term $\left(K^{\beta\alpha\rho\lambda}F_{\beta\alpha}F_{\rho\lambda}\right)$, which is CPT-even, with $K^{\beta\alpha\rho\lambda}$ being the LV tensor coefficient. An interesting study on the electrostatic and magnetostatic features associated with this term was performed by Bailey & Kostelecky [12], which have used the Green function techniques to obtain

the classical solutions for the 4-potential vector in vacuum and in a material medium. Such solutions revealed the interconnection existing between the electric and magnetic sectors in this theory. Other studies involving LV in electrodynamical models are also known in literature [13].

Lorentz violation in the presence of the Higgs sector has also been examined, with the purpose of establishing upper bounds on the associate breaking parameters and studying the Nambu-Goldstone modes [14]. An investigation of the Higgs sector in the context of the MCFJ model was accomplished as well [15]. The resulting Mawell-Carroll-Field-Jackiw electrodynamics with the Proca mass term - Maxwell-Carroll-Field-Jackiw-Proca (MCFJ-Proca) electrodynamics, has had its consistency examined, exhibiting an outcome similar to that of the MCFJ model, that is, the causality and unitarity are assured for a purely spacelike background whereas are spoiled for a purely timelike background. It is still worthy to mention that several properties of the MCFJ electrodynamics were already addressed in a low-dimension space-time. Indeed, such model was properly undergone to a dimensional reduction to (1+2) dimensions, yielding a Maxwell-Chern-Simons (MCS) electrodynamics coupled to a massless Klein-Gordon field (stemming from the previous $A^{(3)}$ component) and a two-dimensional LV background. The consistency of this planar model was examined, revealing a model totally causal, stable and unitary [16]. The static classical solutions of this planar model were determined for a point-like charge [17], revealing the background effects on the electric and magnetic sectors of the MCS electrodynamics. A similar study was also performed for the case of the Maxwell-Carroll-Field-Jackiw electrodynamics with Higgs sector. Such a model was dimensionally reduced to (1+2) dimension as well, having its consistency and classical solutions properly examined [18].

Such detailed investigations on the classical solutions in (1+2) dimensions have not counterpart in the original models [10],[15], defined in (1+3) dimensions. Hence, the purpose of the present work is to study the classical solutions of the MCFJ and MCFJ-Proca models for both purely timelike and purely spacelike backgrounds, for static and stationary sources. The starting point in both cases is the evaluation of the Green function for the tensor equation for the four-potential A^μ, which provides Fourier expressions for the scalar and vector potential in the momentum space. The Fourier transforms of such relations lead to the classical solutions for these potentials, which yield the solutions for the field strengths. In the purely timelike case, the stationary Green function is evaluated. It is observed that the electric field presents an exponentially decaying behavior, independent of the background, equal to the usual Maxwell-Proca result. The magnetic field is null for a static charge and exhibits an intricate oscillating behavior for a stationary moving charge. The limit of a vanishing Proca mass yields the stationary MCFJ solutions. For the case of a purely spacelike background, no exact Fourier transforms for the potentials are obtained. The integrals are then performed under the approximation of a small background ($\mathbf{v}^2 \ll M_A^2$), and the stationary Green function carried out. The electric and magnetic field strengths, evaluated at the $\mathbf{v}^2$-order, exhibit an exponentially decaying behavior. In the limit of a vanishing background, the results recover the usual Maxwell-Proca solution.

This paper is outlined as follows. In Sec. II, it is shown a brief presentation of the basic aspects of the classical MCFJ-Proca model, including equations of motion, energy-momentum tensor, and classical wave equations. In Sec. III, we proceed with the evaluation of the Green function associated with the tensor equation for the four-potential. The stationary Green function is computed explicitly. Expressions for the scalar and vector potentials are derived, which provides explicit solutions for the electric and magnetic field strengths. This is done both for a purely timelike and spacelike background configuration.

II. THE CARROLL-FIELD-JACKIW ELECTRODYNAMICS WITH PROCA MASS

The starting point is the Carroll-Field-Jackiw-Proca Lagrangian, written in (1+3) dimensions:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\nu}F^{\alpha\nu} + \frac{1}{2}M_A^2 A_\alpha A^\alpha - \frac{1}{4}\varepsilon^{\beta\alpha\rho\lambda}V_\beta A_\alpha F_{\rho\lambda} + J^\alpha A_\alpha, \tag{1}$$

with $V^\alpha = (\mathrm{v}_0, \mathbf{v})$ being the fixed background responsible for Lorentz-violation in the gauge sector. Such a model was by first considered in ref. [15], where the Proca mass stems from a Higgs scalar sector. The gauge propagator was evaluated and its consistency was analyzed. It was then shown that this model is unitary just for a spacelike background while it presents ghost states for a timelike or lightlike background. The Euler-Lagrange

equation leads to the modified Maxwell equation,

$$\partial_\nu F^{\nu\alpha} + M_A^2 A^\alpha + V_\beta \tilde{F}^{\alpha\beta} = -J^\alpha, \tag{2}$$

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0, \tag{3}$$

where $\tilde{F}^{\alpha\beta} = \dfrac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$ is the dual tensor, with the convention $\epsilon^{0123} = +1$. From eq. (2), we obtain $M_A^2 \partial_\alpha A^\alpha = -\partial_\alpha J^\alpha$. Considering current conservation ($\partial_\alpha J^\alpha = 0$), the Lorentz gauge ($\partial_\alpha A^\alpha = 0$) appears as an implied condition. The energy and momentum storaged by the electromagnetic field may be taken by the energy-momentum tensor:

$$\Theta^{\alpha\beta} = -F^{\alpha\nu}F^\beta_{\ \nu} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\epsilon^{\alpha\nu\lambda\rho}V^\beta A_\nu F_{\lambda\rho} - \frac{1}{2}g^{\alpha\beta}M_A^2 A_\lambda A^\lambda$$

Once this theory is invariant under space-time translations, the energy-momentum tensor is conserved ($\partial_\alpha \Theta^{\alpha\beta} = 0$) in the absence of sources. This tensor can not be turned symmetric as a consequence of Lorentz-violation. The energy density is the written as

$$\Theta^{00} = \frac{1}{2}[\mathbf{E}^2 + \mathbf{B}^2 + \mathbf{v}_0\mathbf{B}\cdot\mathbf{A} - M_A^2(A_0^2 - \mathbf{A}\cdot\mathbf{A})].$$

This expression reveals that the energy is not positive definite due to the term $\mathbf{v}_0\mathbf{B}\cdot\mathbf{A}$, which may be negative.

The motion equations (2, 3) are explicitly written as the modified Maxwell equations:

$$\nabla\cdot\mathbf{E} + M_A^2 A^0 + \mathbf{v}\cdot\mathbf{B} = -\rho, \tag{4}$$

$$\nabla\times\mathbf{B} - \partial_t\mathbf{E} - \mathbf{v}\times\mathbf{E} = -\mathbf{v}_0\mathbf{B} - M_A^2\mathbf{A} - \mathbf{j}, \tag{5}$$

$$\nabla\times\mathbf{E} = -\partial_t\mathbf{B}, \tag{6}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{7}$$

Manipulating these relations, wave equations for field strengths are readily attained:

$$\left(\Box + M_A^2\right)\mathbf{B} + \mathbf{v}_0\left(\nabla\times\mathbf{B}\right) = \nabla\times(\mathbf{v}\times\mathbf{E}) - \nabla\times\mathbf{j}, \tag{8}$$

$$\left(\Box + M_A^2\right)\mathbf{E} + \partial_t\left(\mathbf{v}\times\mathbf{E}\right) = \nabla(\mathbf{v}\cdot\mathbf{B}) + \mathbf{v}_0\partial_t\mathbf{B} + \nabla\rho + \partial_t\mathbf{j}. \tag{9}$$

Wave equations can be also written for the four-potential:

$$\left[\left(\Box + M_A^2\right)g^{\mu\nu} + \epsilon^{\mu\nu\kappa\lambda}V_\kappa\partial_\lambda\right]A_\nu = -J^\mu, \tag{10}$$

whose scalar and vector components are:

$$\left(\Box + M_A^2\right)A^0 + \mathbf{v}\cdot\mathbf{B} = -\rho, \tag{11}$$

$$\left(\Box + M_A^2\right)\mathbf{A} + \mathbf{v}_0\mathbf{B} - \mathbf{v}\times\mathbf{E} = -\mathbf{j}. \tag{12}$$

These equations reveal a remarkable feature of the CFJ electrodynamics: the electric and magnetic sectors become entwined for the case the background presents a non-null space component ($\mathbf{v} \neq \mathbf{0}$). In this situation, the magnetic field strength contributes for the determination of the scalar potential while the electric field strength affects the vector potential solution. This means that charge and current densities both contribute to electric and magnetic field solutions, so that a static charge originates both electric and magnetic field strengths and a stationary current yields both magnetic and electric fields. A similar mixing of the electric and magnetic sectors is also reported in the context of the electrodynamics related to the term $K^{\beta\alpha\rho\lambda}F_{\beta\alpha}F_{\rho\lambda}$ (see ref. [12]). For a purely timelike background, $V^\mu = (\mathbf{v}_0, \mathbf{0})$, the potential equations decouple and the sector entanglement ceases, recovering the usual uncoupled electromagnetic behavior.

III.   SOLUTION BY THE GREEN METHOD

A complete solution for the potentials can be obtained by the Green method. The implementation of the Green method begins by writing the 4-potential and the 4-current as Fourier transforms in momentum space:

$$A_\mu(r) = \int \frac{d^4 p}{(2\pi)^4} \tilde{A}_\mu(p) \exp(-ip \cdot r), \tag{13}$$

$$J^\mu(r) = \int \frac{d^4 p}{(2\pi)^4} \tilde{J}^\mu(p) \exp(-ip \cdot r), \tag{14}$$

Such expressions must be replaced in eq. (10), providing:

$$D^{\mu\nu} \tilde{A}_\nu(p) = \tilde{J}^\mu(p), \tag{15}$$

The tensor operator $D^{\nu\beta}$ is fully written as

$$D^{\mu\nu} = (p^2 - M_A^2) g^{\mu\nu} + i\varepsilon^{\mu\nu\alpha\kappa} V_\alpha p_\kappa, \tag{16}$$

Its determinant is

$$D = (p^2 - M_A^2) \left[(p \cdot V)^2 - p^2 V^2 - (p^2 - M_A^2)^2 \right], \tag{17}$$

whose zeros play a relevant role in the investigation of the spectrum of the theory, as suitably shown in ref. [15].

The solution for the 4-potential can be constructed in terms of the inverse tensor operator, $[D_{\mu\nu}]^{-1}$, that is:

$$\tilde{A}_\mu(p) = \left[D^{-1} \right]_{\mu\nu} \tilde{J}^\nu(p). \tag{18}$$

$$\left[D^{-1} \right]_{\mu\nu} = \frac{1}{Q} \left\{ (p^2 - M_A^2) g_{\mu\nu} + \frac{V^2}{p^2 - M_A^2} p_\mu p_\nu - i\epsilon_{\mu\nu\rho\sigma} V^\rho p^\sigma + \frac{p^2}{p^2 - M_A^2} V_\mu V_\nu - \frac{(V \cdot p)}{p^2 - M_A^2} (V_\mu p_\nu + V_\nu p_\mu) \right\}, \tag{19}$$

with

$$Q = (p^2 - M_A^2)^2 + V^2 p^2 - (V \cdot p)^2. \tag{20}$$

It is important to point out that the inverse tensor $[D_{\mu\nu}]^{-1}$ is not equal to the photon sector propagator obtained in ref. [15]. Now, we shall particularize the tensor $D^{\nu\beta}$ and the associated solutions for the cases the LV background is purely timelike or purely spacelike.

A.   Solution for a purely timelike background

For a purely timelike background, $V^\alpha = (\mathbf{v}_0, \mathbf{0})$, the components of the inverse tensor operator, $[D_{\nu\beta}]^{-1}$, takes the form

$$\left[D^{-1} \right]_{00}(p) = \frac{1}{p^2 - M_A^2}, \qquad \left[D^{-1} \right]_{0k} = \left[D^{-1} \right]_{k0} = 0, \tag{21}$$

$$\left[D^{-1} \right]_{jk}(p) = \frac{1}{\Delta} \left\{ -(p^2 - M_A^2) \delta_{jk} + \frac{\mathbf{v}_0^2}{p^2 - M_A^2} p_j p_k - i\epsilon_{jkl} \mathbf{v}_0 p_l \right\}, \tag{22}$$

where $\epsilon_{123} = 1$ and Δ is

$$\Delta = (p^2 - M_A^2)^2 - \mathbf{v}_0^2 \mathbf{p}^2. \tag{23}$$

We are interested in stationary solution of the four-potential, thus, setting $p_0 = 0$ the components of the tensor $[D_{\nu\beta}]^{-1}$ read as

$$\left[\bar{D}^{-1}\right]_{00}(\mathbf{p}) = -\frac{1}{\mathbf{p}^2 + M_A^2} \quad , \qquad \left[\bar{D}^{-1}\right]_{0k} = \left[\bar{D}^{-1}\right]_{k0} = 0, \tag{24}$$

$$\left[\bar{D}^{-1}\right]_{jk}(\mathbf{p}) = \frac{1}{\bar{\Delta}}\left\{\left(\mathbf{p}^2 + M_A^2\right)\delta_{jk} - \frac{\mathbf{v}_0^2}{\mathbf{p}^2 + M_A^2}p_j p_k - i\epsilon_{jkl}\mathbf{v}_0 p_l\right\}, \tag{25}$$

with

$$\bar{\Delta} = \left(\mathbf{p}^2 + M_A^2\right)^2 - \mathbf{v}_0^2\mathbf{p}^2. \tag{26}$$

We should now write a general expression for the four-potential by the Green method taking a non-null current density. In this sense, the four-potential is read as

$$A_\mu(\mathbf{r}) = \int d^3\mathbf{r}' G_{\mu\nu}(\mathbf{r} - \mathbf{r}')J^\nu(\mathbf{r}'), \tag{27}$$

where $G_{\mu\nu}(\mathbf{r} - \mathbf{r}')$ is the Green's functions here written in terms of the inverse tensor $\left[\bar{D}^{-1}\right]_{\mu\nu}$:

$$G_{\mu\nu}(\mathbf{r} - \mathbf{r}') = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left[\bar{D}^{-1}\right]_{\mu\nu}(\mathbf{p})\ e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{28}$$

From the matrix $\left[\bar{D}^{-1}\right]_{\mu\nu}(\mathbf{p})$, we can write straightforwardly the components of the Green function in terms of Fourier integrals:

$$G_{00}(\mathbf{R}) = -I_0(\mathbf{R}), \tag{29}$$

$$G_{0k}(\mathbf{R}) = G_{k0}(\mathbf{R}) = 0, \tag{30}$$

$$G_{jk}(\mathbf{R}) = \delta_{jk}I_1(\mathbf{R}) + \mathbf{v}_0^2\partial_j\partial_k I_2(\mathbf{R}) - \mathbf{v}_0\epsilon_{jkl}\partial_l I_3(\mathbf{R}), \tag{31}$$

where the integrals $I_i(\mathbf{R})$ are defined bellow:

$$I_0(\mathbf{R}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{R}}}{\mathbf{p}^2 + M_A^2} = \frac{1}{4\pi}\frac{e^{-M_A R}}{R}, \tag{32}$$

$$I_1(\mathbf{R}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\left(\mathbf{p}^2 + M_A^2\right)e^{i\mathbf{p}\cdot\mathbf{R}}}{\left(\mathbf{p}^2 - m_+^2\right)\left(\mathbf{p}^2 - m_-^2\right)}, \tag{33}$$

$$I_2(\mathbf{R}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{R}}}{\left(\mathbf{p}^2 + M_A^2\right)\left(\mathbf{p}^2 - m_+^2\right)\left(\mathbf{p}^2 - m_-^2\right)}, \tag{34}$$

$$I_3(\mathbf{R}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{R}}}{\left(\mathbf{p}^2 - m_+^2\right)\left(\mathbf{p}^2 - m_-^2\right)}. \tag{35}$$

where $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$. In order to solve these integrals, we first have factorized the denominator as $\left(\mathbf{p}^2 + M_A^2\right)^2 - \mathbf{v}_0^2\mathbf{p}^2 = \left(\mathbf{p}^2 - m_+^2\right)\left(\mathbf{p}^2 - m_-^2\right)$. It is very important to remark that the massive poles $m_\pm^2$,

$$m_\pm^2 = \frac{1}{2}\left[\mathbf{v}_0^2 - 2M_A^2 \pm \mathbf{v}_0\sqrt{\mathbf{v}_0^2 - 4M_A^2}\right], \tag{36}$$

are positive under the condition $\mathbf{v}_0^2 \geqslant 4M_A^2$. This fact is responsible for the oscillatory character of the magnetic sector solutions to be achieved for this model. The remaining three integrals can be solved in the complex plane, yielding:

$$I_1(\mathbf{R}) = \frac{1}{4\pi R}\left[\frac{a_+}{a}\cos(m_+ R) - \frac{a_-}{a}\cos(m_- R)\right], \tag{37}$$

$$I_2(\mathbf{R}) = \frac{1}{4\pi R}\left[\frac{e^{-M_A R}}{a_+ a_-} + \frac{\cos(m_+ R)}{a a_+} - \frac{\cos(m_- R)}{a a_-}\right], \tag{38}$$

$$I_3(\mathbf{R}) = \frac{1}{4\pi R}\frac{1}{a}\left[\cos m_+ R - \cos m_- R\right], \tag{39}$$

with

$$a = \mathbf{v}_0 \sqrt{\mathbf{v}_0^2 - 4M_A^2}, \qquad a_\pm = \frac{1}{2} \left[\mathbf{v}_0^2 \pm \mathbf{v}_0 \sqrt{\mathbf{v}_0^2 - 4M_A^2} \right]. \tag{40}$$

Replacing these results in eq. (31), the Green function $G_{jk}(\mathbf{r} - \mathbf{r}')$ is finally obtained:

$$
\begin{aligned}
G_{jk}(\mathbf{R}) = {}& \frac{1}{4\pi R} \delta_{jk} \left[\frac{a_+}{a} \cos(m_+ R) - \frac{a_-}{a} \cos(m_- R) \right] \\
& - \frac{1}{4\pi R^3} \frac{e^{-M_A R}}{M_A^2} \left[\Delta_{jk}(1 + R M_A) - M_A^2 R_j R_k \right] \\
& - \frac{1}{4\pi R^3} \mathbf{v}_0^2 \left[\frac{(\Delta_{jk} + m_+^2 R_j R_k)}{a a_+} \cos m_+ R - \frac{(\Delta_{jk} + m_-^2 R_j R_k)}{a a_-} \cos m_- R \right] \\
& - \frac{1}{4\pi R^2} \mathbf{v}_0^2 \Delta_{jk} \left[\frac{m_+}{a a_+} \sin m_+ R - \frac{m_-}{a a_-} \sin m_- R \right] \\
& + \frac{\mathbf{v}_0 \epsilon_{jkl} R_l}{a R^3} \left[(\cos m_+ R - \cos m_- R) + R(m_+ \sin m_+ R - m_- \sin m_- R) \right].
\end{aligned}
\tag{41}
$$

where $\Delta_{jk} = \left(\delta_{kj} - 3 R_j R_k / R^2 \right)$. This Green function can be simplified to its MCFJ counterpart taking the limit $(M_A \to 0)$:

$$
\begin{aligned}
G_{jk}(\mathbf{R}) = {}& \frac{1}{4\pi R} \delta_{jk} \cos(\mathbf{v}_0 R) + \frac{1}{4\pi R^3} \Delta_{jk} \left[\frac{R^2}{2} - \frac{R}{\mathbf{v}_0} \sin(\mathbf{v}_0 R) + \frac{1 - \cos(\mathbf{v}_0 R)}{\mathbf{v}_0^2} \right] \\
& + \frac{1}{4\pi R^3} R_j R_k \left[1 - \cos(\mathbf{v}_0 R) \right] + \frac{\epsilon_{jkl} R_l}{R^3} \left[R \sin \mathbf{v}_0 R - \frac{1 - \cos \mathbf{v}_0 R}{\mathbf{v}_0} \right].
\end{aligned}
\tag{42}
$$

It is easy to verify that such result in the Lorentz symmetric limit $(\mathbf{v}_0 \to 0)$ rescues the pure Maxwell result

$$G_{jk}(\mathbf{R}) = \frac{1}{4\pi R} \delta_{jk}. \tag{43}$$

Turning back to the issue of calculating explicit classical solutions, we address the solution for the scalar potential. Regarding eq. (27), and the density current for a point-like charge, $J_0(\mathbf{r}) = e\delta(\mathbf{r})$, the following expression is obtained for the scalar potential

$$A_0(\mathbf{r}) = -\frac{e}{4\pi} \frac{e^{-M_A r}}{r}. \tag{44}$$

where $r = ||\mathbf{r}||$. The electric field may be easily evaluated from the scalar potential $(\mathbf{E} = -\nabla A_0)$, exhibiting an exponentially decaying solution as well:

$$\mathbf{E}(\mathbf{r}) = -\frac{e}{4\pi} \left(\frac{M_A}{r} + \frac{1}{r^2} \right) e^{-M_A r} \, \hat{\mathbf{r}}. \tag{45}$$

It is interesting to note that this is the same result obtained for the electric field of the Maxwell-Proca Lagrangian (without Lorentz violation, $V^\mu = 0$). This means that the Lorentz-violating background, when coupled to the gauge field as in Lagrangian (1), does not alter the electrostatic sector. Analysis of Maxwell equations (4,5,6) in the static regime reveals that this scenario remains true even in the presence of a non-null current. Hence, the scalar potential and electric field achieved here are the same for a point-like charge in uniform motion (stationary solution), once the current $\mathbf{J}$ does not contribute to A_0. In the absence of the Proca mass, the solutions (44, 45) reduce to the CFJ ones, which coincide with the Coulombian result:

$$A_0(\mathbf{r}) = -\frac{e}{4\pi} \frac{1}{r}, \qquad \mathbf{E}(\mathbf{r}) = -\frac{e}{4\pi} \frac{1}{r^2} \, \hat{\mathbf{r}}. \tag{46}$$

The fixed background, therefore, does to not induce any effect on the electric field solution of the MCFJ model, as well. This fact deserves to be compared with the scenario of the dimensionally reduced version of this model, studied in ref. [17]. In such work, it was shown that the purely timelike background alters the character of the classical Maxwell-Chern-Simons electric solution, turning the usual screened Bessel-like solution into an unscreened $1/r$ electric field.

The solution for vector potential can be found by the same procedure. From the inverse tensor $[D_{\nu\beta}]^{-1}$, we obtain the expression for the vector potential Fourier transform:

$$\tilde{\mathbf{A}}_i\left(\mathbf{p}\right) = -\frac{\left(\mathbf{p}^2 + M_A^2\right)\tilde{J}_i + i\mathbf{v}_0\left(\mathbf{p}\times\tilde{\mathbf{J}}\right)_i}{\left[\left(\mathbf{p}^2 + M_A^2\right)^2 - \mathbf{v}_0^2\mathbf{p}^2\right]}, \tag{47}$$

where it was considered that $\tilde{\mathbf{J}}\cdot\mathbf{p} = 0$, as a consequence of current conservation $(p_\alpha\tilde{J}^\alpha = 0)$. In this case, we take the current associated with a point-like charge in uniform motion with velocity $\mathbf{u}$,

$$\mathbf{J}\left(\mathbf{r}\right) = e\mathbf{u}\delta\left(\mathbf{r}\right), \tag{48}$$

In the momentum space, $\tilde{\mathbf{J}}\left(\mathbf{p}\right) = e\mathbf{u}$. The vector potential, written as the Fourier transform of eq. (47), may be compactly expressed in terms of the integrals (37,39), so that:

$$\mathbf{A}\left(\mathbf{r}\right) = -e\mathbf{u}I_1\left(r\right) + e\mathbf{v}_0\ \mathbf{u}\times\nabla I_3\left(r\right). \tag{49}$$

Considering the results already obtained, the vector potential takes the form:

$$\mathbf{A}\left(\mathbf{r}\right) = -\frac{e}{4\pi}\left\{\frac{1}{r}\left[A_+\cos\left(m_+r\right) - A_-\cos\left(m_-r\right)\right]\ \mathbf{u}\right.$$
$$\left. + \frac{\mathbf{v}_0}{ar^3}\left[\left(\cos m_+r - \cos m_-r\right) + r\left(m_+\sin m_+r - m_-\sin m_-r\right)\right]\ \mathbf{u}\times\mathbf{r}\right\}, \tag{50}$$

where $A_\pm = a_\pm/a$. For the case of a vanishing Proca mass $(M_A\to 0)$, this expression is reduced to the MCFJ vector solution:

$$\mathbf{A}\left(\mathbf{r}\right) = -\frac{e}{4\pi}\left\{\frac{\cos\left(\mathbf{v}_0r\right)}{r}\mathbf{u} + \frac{1}{r^3}\left[\frac{\cos(\mathbf{v}_0r) - 1}{\mathbf{v}_0} + r\sin(\mathbf{v}_0r)\right]\mathbf{u}\times\mathbf{r}\right\}. \tag{51}$$

It is interesting to note that oscillating solutions are obtained in both cases, and the Proca mass is not a factor able to annihilate such a behavior. For the case of a point-like static charge, the potential vector and the magnetic field strength are null, once it depends only on the current source $(\mathbf{j}\neq 0)$. Hence, for a point-like charge in uniform motion, $\mathbf{J}\left(\mathbf{r}\right) = e\mathbf{u}\delta\left(\mathbf{r}\right)$, the magnetic field becomes non-null. It can be derived from the vector potential $(\mathbf{B} = \nabla\times\mathbf{A})$, yielding:

$$\mathbf{B}\left(\mathbf{r}\right) = -\frac{e}{4\pi r^3}\left\{\left[A_+\cos\left(m_+r\right) - A_-\cos\left(m_-r\right) + A_+m_+r\sin\left(m_+r\right) - A_-m_-r\sin\left(m_-r\right)\right]\left(\mathbf{u}\times\mathbf{r}\right)\right.$$
$$\left. - \frac{\mathbf{v}_0}{a}\left[\left(1 - m_+^2r^2\right)\cos m_+r - \left(1 - m_-^2r^2\right)\cos m_-r + m_+r\sin\left(m_+r\right) - m_-r\sin\left(m_-r\right)\right]\mathbf{u}\right\}. \tag{52}$$

An additional contribution proportional to $(\mathbf{u}\cdot\mathbf{r})\mathbf{r}$ also appears in the above expression, but it is was discarded because the scalar product $\mathbf{u}\cdot\mathbf{r}$ is null. This result stems from the condition $\nabla\cdot\mathbf{A} = \mathbf{0}$, a consequence of the external current conservation in the stationary regime. This magnetic field solution exhibits two components: one in the direction the velocity $\mathbf{u}$, and another orthogonal to the plane defined by $\mathbf{u}$ and $\mathbf{r}$. This magnetic field exhibits a decaying $1/r^2$ behavior both near the origin $(r\to 0)$ and asymptotically. In the limit of a vanishing Proca mass $(M_A\to 0)$, this expression is reduced to the MCFJ solution:

$$\mathbf{B}\left(\mathbf{r}\right) = -\frac{e}{4\pi r^3}\left\{\left[\cos(\mathbf{v}_0r) + \mathbf{v}_0r\sin(\mathbf{v}_0r)\right]\left(\mathbf{u}\times\mathbf{r}\right) - \left[\frac{1 - \mathbf{v}_0^2r^2}{\mathbf{v}_0}\cos(\mathbf{v}_0r) - \frac{1}{\mathbf{v}_0} + r\sin(\mathbf{v}_0r)\right]\mathbf{u}\right\}. \tag{53}$$

This solution, similarly to the MCFJ-Proca one, is also decomposed in terms of two orthogonal directions, $\mathbf{u}$ and $\mathbf{u} \times \mathbf{r}$. The fact that the MCFJ magnetic field exhibits an oscillating behavior, associated with the $1/r^2$ behavior of the MCFJ electric field, is compatible with the emission of Cerenkov radiation by a point-like charge in uniform motion [9]. At the same way, the exponentially decaying behavior of eq. (45) puts in evidence that no Cerenkov radiation can be emitted by a stationary charge in the framework of the MCFJ-Proca electrodynamics, once one condition to have radiation is that the fields should present a non null asymptotic behavior [9].

Another regime in which such solutions shall be investigated is the one of a vanishing LV background ($\mathbf{v}_0 \rightarrow 0$), which should lead back to the usual Maxwell-Proca electrodynamics. This limit, however, can not be implemented directly on the MCFJ-Proca expressions of eqs. (50,52), since these solutions were derived under the condition $\mathbf{v}_0^2 \geqslant 4M_A^2$, which assures that the poles (36) be real and positive definite. A way to avoid this complication is to implement the limit $\mathbf{v}_0 \rightarrow 0$ on eqs. (51,53). The results obtained,

$$\mathbf{A}\left(\mathbf{r}\right) = -\frac{e}{4\pi r} \; \mathbf{u}, \quad \mathbf{B}\left(\mathbf{r}\right) = -\frac{e}{4\pi r^3} \; \mathbf{u} \times \mathbf{r}, \tag{54}$$

nonetheless, do not recover the correct Maxwell-Proca behavior,

$$\mathbf{A}\left(\mathbf{r}\right) = -\frac{e}{4\pi r} e^{-M_A r} \; \mathbf{u}, \quad \mathbf{B}\left(\mathbf{r}\right) = -\frac{e}{4\pi r^3} \left[1 + M_A r\right] e^{-M_A r} \; \mathbf{u} \times \mathbf{r}, \tag{55}$$

attained from $\tilde{\mathbf{A}}_i\left(p\right) = -\tilde{J}_i / \left(\mathbf{p}^2 + M_A^2\right)$ [eq. (47) taken in the limit $\mathbf{v}_0 \rightarrow 0$]. This apparent non equivalence simply shows that Maxwell-Proca behavior may not be found as the limit $\mathbf{v}_0 \rightarrow 0$ of the MCFJ-Proca solutions. This is ascribed to the structure of poles appearing in the MCFJ model, $1/\left(\mathbf{p}^2 + m^2\right)$, associated with an exponentially decaying behavior, in contrast with the MCFJ-Proca pole structure, $1/\left(\mathbf{p}^2 - m^2\right)$, related to an oscillating behavior. In this way, we see that the background turns the exponentially decaying behavior of the Maxwell-Proca model into a oscillating solution that goes as $1/r^2$ far from the origin. This is true in both MCFJ and MCFJ-Proca models.
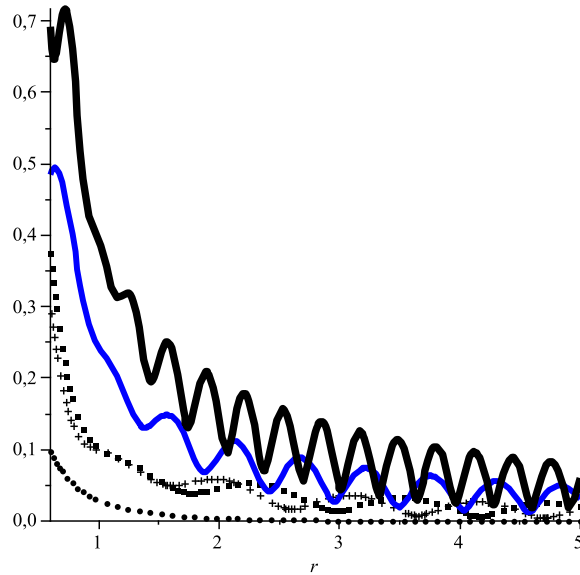

FIG. 1: Simultaneous plot for the magnetic field in the radial interval $0.4 < r < 4$ of the following models: Maxwell-Proca (for $M_A = 1$) - circle dotted line; MCFJ - cross dotted line (for $v_0 = 3, M_A = 0$); MCFJ-Proca - box dotted line (for $v_0 = 3, M_A = 1$), MCFJ-Proca - intermediary continuous line (for $v_0 = 6, M_A = 1$) , MCFJ-Proca - thicker continuous line (for $v_0 = 10, M_A = 1$), and $u = 0.5$ in all cases.

The graph of Fig.1 depicts a simultaneous plot of the Maxwell-Proca, MCFJ and MCFJ-Proca magnetic fields in the case the velocity $\mathbf{u}$ is orthogonal to the vector $\mathbf{r}$. Such a graph shows clearly that the MCFJ-Proca

solution deviates substantially from the Maxwell-Proca behavior mainly due to the presence of oscillation. This deviation increases with the background magnitude: the larger is the background, more pronounced is the amplitude and the frequency of such oscillations. For a background not so large in comparison with the Proca mass ($\mathbf{v}_0 \gtrsim M_A$), the MCFJ e MCFJ-Proca solutions are close from each other. Keeping M_A constant while $\mathbf{v}_0$ increases, it occurs that these solutions become different. This behavior is confirmed in the graph of Fig.2, where it is shown a comparison between the MCFJ and MCFJ-Proca magnetic solutions for equal values of $\mathbf{v}_0$ (for the case the velocity $\mathbf{u}$ is orthogonal to the vector $\mathbf{r}$). In general, these solutions are nearly coincident, but deviate from each other for larger M_A magnitudes.
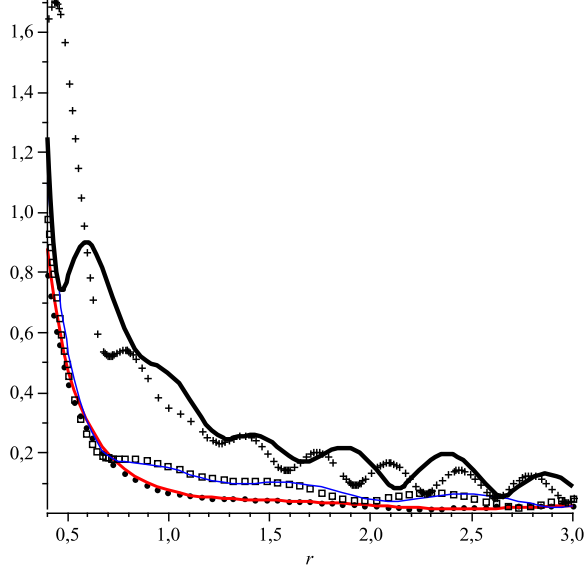

FIG. 2: Simultaneous plot of the magnetic fields for the MCFJ (dotted curves - $M_A = 0$) and MCFJ-Proca (continuous curves) models in the radial interval $0.4 < r < 3$. The lower two curves - point dotted and continuous superposed line - correspond to ($\mathbf{v}_0 = 2, M_A = 0$) and ($\mathbf{v}_0 = 2, M_A = 0.5$). For the two intermediary curves - box dotted and continuous thin line: ($\mathbf{v}_0 = 4, M_A = 0,$) and ($\mathbf{v}_0 = 4, M_A = 1$). For the two upper curves - cross dotted and thicker continuous line: ($\mathbf{v}_0 = 9, M_A = 0$) and ($\mathbf{v}_0 = 9, M_A = 4$). It was used $u = 0.5$ in all graphs.

In order to find a general expression for the magnetic field strength, a direct way consists in starting from wave equation (8), which in the stationary regime and for a purely timelike background is simplified to:

$$\left[\nabla^2(\nabla^2 - 2M_A^2 + \mathbf{v}_0^2) + M_A^4\right]\mathbf{B} = -M_A^2\left(\nabla \times \mathbf{j}\right) - \mathbf{v}_0\nabla^2\mathbf{j} + \nabla^2\left(\nabla \times \mathbf{j}\right). \tag{56}$$

The corresponding Green function for this equation is $G\left(\mathbf{r} - \mathbf{r}'\right) = \int \dfrac{d^3\mathbf{p}}{(2\pi)^3}\tilde{G}\left(\mathbf{p}\right)e^{i\mathbf{p}\cdot\left(\mathbf{r}-\mathbf{r}'\right)}$, with $\tilde{G}\left(p\right) = \left[\left(\mathbf{p}^2 - m_+^2\right)\left(\mathbf{p}^2 - m_-^2\right)\right]^{-1}$, and the poles $m_\pm^2$ being given by eq. (36). In true, this Green function was already evaluated, corresponding to the result of integral I_3, so that the magnetic field strength is:

$$\mathbf{B}\left(\mathbf{r}\right) = \frac{1}{4\pi a}\int \frac{1}{R}\left[\cos m_+R - \cos m_-R\right]\left[-M_A^2\nabla \times \mathbf{j}\left(\mathbf{r}'\right) - \mathbf{v}_0\nabla^2\mathbf{j}\left(\mathbf{r}'\right) + \nabla^2\nabla \times \mathbf{j}\left(\mathbf{r}'\right)\right]d^3\mathbf{r}'. \tag{57}$$

This outcome differs from the MCFJ one presented in ref. [10], which exhibits an exponentially decaying behavior. For the case of the pure CFJ model ($M_A \to 0$), we get:

$$\mathbf{B}\left(\mathbf{r}\right) = \frac{1}{4\pi\mathbf{v}_0^2}\int d^3\mathbf{r}\left(\frac{1 - \cos(\mathbf{v}_0 R)}{R}\right)\nabla^2\left[\mathbf{v}_0\mathbf{j}\left(\mathbf{r}'\right) - \nabla \times \mathbf{j}\left(\mathbf{r}'\right)\right], \tag{58}$$

with $R = ||\mathbf{r} - \mathbf{r}'||$. Using the Green theorem, $\int_V \left[\phi\nabla^2\chi - \chi\nabla^2\phi\right] d^3r = \int_{\partial R} \left[\phi\partial\chi/\partial n - \chi\partial\phi/\partial n\right] dS$, and considering that the current and its derivatives are null on a very distant surface, then eq. (58) is rewritten as:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi} \int \left(\frac{\cos(\mathbf{v}_0 R)}{R}\right) \left[\mathbf{v}_0 \mathbf{j}(\mathbf{r}') - \nabla \times \mathbf{j}(\mathbf{r}')\right] \ d^3\mathbf{r}. \tag{59}$$

where it was used $\nabla^2[(1 - \cos(\mathbf{v}_0 R))/R] = \mathbf{v}_0^2 \cos(\mathbf{v}_0 R)/R$. This result is equal to the one of ref. [10] apart from a global signal stemming from our definition of the external current vector $\mathbf{j}(\mathbf{r})$.

B.   Solution for a purely spacelike background

The case the background is purely spacelike, $V^\alpha = (0, \mathbf{v})$, becomes particularly interesting when we consider the model of Lagrangian (1) under the perspective of its physical consistency. In fact, it is known that this model exhibits full consistency (stability, causality, unitarity) only for a spacelike background. Henceforth, this is a good reason to study the corresponding classical solutions for a purely spacelike background. We begin writing the elements of the inverse matrix (19) for $V^\alpha = (0, \mathbf{v})$:

$$\left[D^{-1}\right]_{00} = \frac{p^2 - M_A^2}{D} - \frac{\mathbf{v}^2}{D\left(p^2 - M_A^2\right)}\left(p_0\right)^2, \tag{60}$$

$$\left[D^{-1}\right]_{0k} = i\epsilon_{kjl}\mathbf{v}_j p_l + \frac{(\mathbf{v}\cdot\mathbf{p})\mathbf{v}_k - \mathbf{v}^2 p_k}{D\left(p^2 - M_A^2\right)}p_0, \tag{61}$$

$$\left[D^{-1}\right]_{k0} = -i\epsilon_{kjl}\mathbf{v}_j p_l + \frac{(\mathbf{v}\cdot\mathbf{p})\mathbf{v}_k - \mathbf{v}^2 p_k}{D\left(p^2 - M_A^2\right)}p_0, \tag{62}$$

$$\left[D^{-1}\right]_{jk} = -\frac{p^2 - M_A^2}{D}\delta_{jk} - \frac{\mathbf{v}^2 p_j p_k - p^2\mathbf{v}_j\mathbf{v}_k - (\mathbf{v}\cdot\mathbf{p})\left(\mathbf{v}_j p_k + \mathbf{v}_k p_j\right)}{D\left(p^2 - M_A^2\right)} + \frac{i\epsilon_{jkl}\mathbf{v}_l p_0}{D}, \tag{63}$$

with $D = \left(p^2 - M_A^2\right)^2 - \mathbf{v}^2 p^2 - (\mathbf{v}\cdot\mathbf{p})^2$.

As we are to study the stationary solutions of the model, we set $p_0 = 0$ in the above equations to get the Green function for the wave equation for the stationary four potential:

$$G_{00}(\mathbf{p}) = -\frac{\mathbf{p}^2 + M_A^2}{R_s}, \qquad G_{0k}(\mathbf{p}) = i\frac{(\mathbf{v}\times\mathbf{p})_k}{R_s} = -G_{k0}, \tag{64}$$

$$G_{jk}(\mathbf{p}) = \frac{\mathbf{p}^2 + M_A^2}{R_s}\delta_{jk} + \frac{\delta_{jk}(\mathbf{v}\times\mathbf{p})\cdot(\mathbf{v}\times\mathbf{p}) - (\mathbf{v}\times\mathbf{p})_j(\mathbf{v}\times\mathbf{p})_k}{\left(\mathbf{p}^2 + M_A^2\right)R_s}, \tag{65}$$

where

$$R_s = \left(\mathbf{p}^2 + M_A^2\right)^2 + \mathbf{p}^2\mathbf{v}^2 - (\mathbf{p}\cdot\mathbf{v})^2. \tag{66}$$

Supposing that θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{v}$, $(\mathbf{p}\cdot\mathbf{v} = |\mathbf{p}||\mathbf{v}|\cos\theta)$, the denominator R_s is read as:

$$R_s = \left(\mathbf{p}^2 + M_A^2\right)^2 + \mathbf{p}^2\mathbf{v}^2\sin^2\theta. \tag{67}$$

Similarly to the timelike case, we can construct the stationary Green function from the inverse matrix (64). In this sense, we write:

$$G_{00}(\mathbf{r} - \mathbf{r}') = -F_2(\mathbf{r} - \mathbf{r}'), \quad G_{0k}(\mathbf{r} - \mathbf{r}') = (\mathbf{v}\times\nabla)_k F_1(\mathbf{r} - \mathbf{r}'), \tag{68}$$

$$G_{jk}(\mathbf{r} - \mathbf{r}') = \delta_{jk}F_2(\mathbf{r} - \mathbf{r}') - \left[\delta_{jk}(\mathbf{v}\times\nabla)\cdot(\mathbf{v}\times\nabla) - (\mathbf{v}\times\nabla)_j(\mathbf{v}\times\nabla)_k\right]F_3(\mathbf{r} - \mathbf{r}') \tag{69}$$

where the Fourier transforms F_i are defined as:

$$F_1(\mathbf{R}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot\mathbf{R}}}{R_s}, \tag{70}$$

$$F_2(\mathbf{R}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{(\mathbf{p}^2 + M_A^2)\, e^{i\mathbf{p}\cdot\mathbf{R}}}{R_s}, \tag{71}$$

$$F_3(\mathbf{R}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot\mathbf{R}}}{(\mathbf{p}^2 + M_A^2)\, R_s}. \tag{72}$$

The main difficult concerning the Fourier transforms (70-72) is that we can not calculate an exact solution for such integrations due to the presence of the angular factor in the denominator R_s. A possible way to circumvent this impossibility is to perform the integration under some approximation. A feasible approximation consists in considering the background small before the Proca mass ($\mathbf{v}^2 \ll M_A^2$), which implies the following second order expansion:

$$\frac{1}{R_s} = \frac{1}{(\mathbf{p}^2 + M_A^2)^2} - \frac{\mathbf{p}^2 \sin^2\theta}{(\mathbf{p}^2 + M_A^2)^4}\mathbf{v}^2 + \mathcal{O}(\mathbf{v}^4). \tag{73}$$

With it, and regarding the special situation in which the background $\mathbf{v}$ and the vector $\mathbf{r}$ are parallel, the following second order solutions are obtained:

$$F_1(R) = \frac{1}{8\pi M_A}\left[1 - \frac{\mathbf{v}^2}{12}\left(\frac{R}{M_A} + \frac{1}{M_A^2}\right)\right]e^{-M_A R}, \tag{74}$$

$$F_2(R) = \frac{1}{4\pi}\left[\frac{1}{R} - \frac{\mathbf{v}^2}{4M_A}\right]e^{-M_A R}, \tag{75}$$

$$F_3(R) = \frac{1}{32\pi}\left[\frac{1}{M_A^3} + \frac{R}{M_A^2} - \frac{\mathbf{v}^2}{24}\left(\frac{3}{M_A^5} + \frac{3R}{M_A^4} + \frac{R^2}{M_A^3}\right)\right]e^{-M_A R} \tag{76}$$

The Green functions are finally written as:

$$G_{00}(\mathbf{R}) = -\frac{1}{4\pi}\left[\frac{1}{R} - \frac{\mathbf{v}^2}{4M_A}\right]e^{-M_A R}, \tag{77}$$

$$G_{0i}(\mathbf{R}) = \frac{1}{8\pi}\left[-\frac{1}{R} + \frac{\mathbf{v}^2}{12M_A}\right]e^{-M_A R}(\mathbf{v} \times \mathbf{R})_i \tag{78}$$

$$G_{jk}(\mathbf{R}) = \frac{1}{4\pi}\left\{\delta_{jk}\left[\frac{1}{R} - \frac{\mathbf{v}^2}{8}\left(R + \frac{1}{M_A}\right) + \frac{1}{8R}(\mathbf{v}\cdot\mathbf{R})^2\right]e^{-M_A R}\right. \tag{79}$$
$$\left. + \frac{1}{8}\left[\frac{\mathbf{v}_j\mathbf{v}_k}{M_A} + \frac{1}{R}(\mathbf{v}\times\mathbf{R})_j(\mathbf{v}\times\mathbf{R})_k\right]e^{-M_A R} + \mathcal{O}(\mathbf{v}^4)\right\}$$

We can now write the solutions for the scalar and vector potential, starting from the corresponding Fourier transforms extracted from eqs. (18,64):

$$\tilde{A}_0(\mathbf{p}) = -\frac{(\mathbf{p}^2 + M_A^2)}{R_s}\tilde{J}_0 + \frac{i(\mathbf{p}\times\mathbf{v})\cdot\tilde{\mathbf{J}}}{R_s} \tag{80}$$

$$\tilde{\mathbf{A}}(\mathbf{p}) = \frac{i(\mathbf{p}\times\mathbf{v})}{R_s}\tilde{J}_0 - \frac{(\mathbf{p}^2 + M_A^2)\tilde{\mathbf{J}}}{R_s} - \frac{(\mathbf{p}\times\mathbf{v})\cdot(\mathbf{p}\times\mathbf{v})\tilde{\mathbf{J}}}{(\mathbf{p}^2 + M_A^2)\,R_s} + (\mathbf{p}\times\mathbf{v})\frac{(\mathbf{p}\times\mathbf{v})\cdot\tilde{\mathbf{J}}}{(\mathbf{p}^2 + M_A^2)\,R_s}. \tag{81}$$

Such equations show clearly that the electric field and the magnetic sectors are now entwined. In fact, a static charge is able to create a magnetic field $\left(\tilde{\mathbf{A}} \neq \mathbf{0}\right)$ as much a stationary current is capable to imply a non-null electric field ($\mathbf{A}_0 \neq \mathbf{0}$). Hence, the electric and magnetic fields coexist simultaneously both for a static charge or a stationary current. Such scenario occurs only for a space-like Lorentz breaking background. We now study the

solutions corresponding to a point-like charge $[J_0(\mathbf{r}) = e\delta(\mathbf{r}), \ \mathbf{J}(\mathbf{r}) = e\mathbf{u}\delta(\mathbf{r})]$. In momentum space, $\tilde{J}_0(p) = e$, $\tilde{J}(p) = e\mathbf{u}$. Working at second order in the background magnitude, we obtain the following expressions for scalar and vector potential as Fourier transform of the expressions (80,81), so that it holds:

$$A_0(\mathbf{r}) = -eF_2(r) + e\left[(\nabla \times \mathbf{v}) \cdot \mathbf{u}\right] F_1(r), \tag{82}$$

$$\mathbf{A}(\mathbf{r}) = e(\nabla \times \mathbf{v}) F_1(r) - eF_2(r) \ \mathbf{u} + e\left[(\nabla \times \mathbf{v}) \cdot (\nabla \times \mathbf{v})\mathbf{u} - (\nabla \times \mathbf{v})\left[(\nabla \times \mathbf{v}) \cdot \mathbf{u}\right]\right] F_3(r) . \tag{83}$$

Taking the expressions (74,76), the solutions for these potentials are readily achieved:

$$A_0(\mathbf{r}) = -\frac{e}{4\pi}\left(\frac{1}{r} - \frac{\mathbf{v}^2}{4M_A}\right) e^{-M_A r}, \tag{84}$$

$$\mathbf{A}(\mathbf{r}) = -\frac{e}{4\pi}\left[\left(\frac{1}{r} - \frac{\mathbf{v}^2}{8M_A}\right)\mathbf{u} + \frac{1}{8M_A}(\mathbf{v} \cdot \mathbf{u})\mathbf{v}\right] e^{-M_A r}, \tag{85}$$

These are the general expressions for the scalar and vector potential in $\mathbf{v}^2$ order taking into account the condition that $\mathbf{v}$ and $\mathbf{r}$ are parallel. It is seen that the vector potential only retain the contributions proportional to the velocity $\mathbf{u}$, while these contributions do not appear in the scalar potential. This means that the current does not contribute to the electric sector and the static charge does not yield magnetic field. The explicit expressions for the electric and magnetic field strength,

$$\mathbf{E}(\mathbf{r}) = -\frac{e}{4\pi}\left(\frac{1}{r^2} + \frac{M_A}{r} - \frac{\mathbf{v}^2}{4}\right) e^{-M_A r} \ \hat{\mathbf{r}}. \tag{86}$$

$$\mathbf{B}(\mathbf{r}) = -\frac{e}{4\pi}\left[\frac{1}{r^2} + \frac{M_A}{r} - \frac{\mathbf{v}^2}{8}\right] e^{-M_A r} \ \mathbf{u} \times \hat{\mathbf{r}}. \tag{87}$$

confirm that this is really the case (at least in the context of the approximations adopted) in $\mathbf{v}^2$ order taking into account the condition that $\mathbf{v}$ and $\mathbf{r}$ are parallel. Near the origin, these solutions present a $1/r^2$ behavior, while far from the origin both fields possess a totally screened behavior, which avoids any attempt to obtain Cerenkov radiation for such background. It is instructive to mention that these expressions provide the Maxwell-Proca solutions in the limit $(\mathbf{v} \to \mathbf{0})$, given by eqs. (45,55). Although the limit $(M_A \to 0)$ could be easily carried out in the above expressions, such a limit has no sense here, once these equations have been derived under the supposition that $(\mathbf{v}^2 \ll M_A^2)$.

IV. CONCLUSION

In this work, we have studied the classical static and stationary solutions of the Lorentz-violating MCFJ-Proca model, while the MCFJ solutions were also obtained as the limit of a vanishing Proca mass $(M_A \to \mathbf{0})$. Starting from the equation of motion for the four-potential, the corresponding Green function was written in a matrix form, which provides explicit expressions for the four-potential in momentum space. In the purely timelike case, it was written the Green function for a stationary configuration. An exponentially decaying solution was attained for the electric sector, which is equal to the Maxwell-Proca solutions. In the limit $(M_A \to \mathbf{0})$, these solutions recover the Coulombian ones, which reveals that the background does not modify the electric sector of both MCFJ-Proca and MCFJ models. This is to be contrasted with the LV low-dimensional scenario of the model of ref. [17], established as the dimensional reduction of the MCFJ model, where the Maxwell-Chern-Simons electric sector was drastically altered by the presence of the timelike background. In concerning the magnetic sector, the background induces a radical modification. In fact, while the Lorentz symmetric Maxwell-Proca model exhibits an exponentially decaying magnetic field strength, the MCFJ-Proca model presents an oscillating magnetic field solution for charges in stationary motion. This change in behavior is ascribed to the sign reversion in the poles of the theory. Taking the limit $(M_A \to \mathbf{0})$, the magnetic solution for a stationary charge in the MCFJ model is obtained, whose oscillating behavior is compatible with the emission of Cerenkov

radiation in such framework, as shown in refs. [9]. The emission of Cerenkov radiation in the context of the MCFJ-Proca model is not allowed, due to the exponentially screened behavior of the electric field.

In the purely spacelike case, the same procedure was employed. The Green function for a stationary scenario was written in terms of Fourier integrals, which were solved in a second order approximation into the regime of a small background $\left(\mathbf{v}^2 \ll M_A^2\right)$. The solutions were derived for a special configuration in which the background is parallel to the position of observation of the fields. It was then verified that the electric and magnetic sectors present exponentially decaying solutions, which recover the Maxwell-Proca behavior in the limit $(\mathbf{v} \to \mathbf{0})$.

In a forthcoming work, we intend to use the Green function techniques to investigate the classical radiation emitted by a point-like charge in uniform linear acceleration and in uniform circular motion (synchrotron radiation). In this case, we shall search for time-dependent solutions for the Green functions which may lead to the calculation of the time-dependent solutions that yield radiation.

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